

GSTC formulation for Maxwell's equations

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Abstract

We revisit the classical zero-thickness Generalized Sheet Transition Conditions (GSTCs) which are a key tool for efficiently designing metafilms able to control the flow of light in a desired way. It is shown that it is more convenient to use an enlarged formulation of the GSTC in which the original metafilm is replaced by GSTCs that exclude the layer from the physical or computational domain. These new “layer” transition conditions have the same form as their “sheet” analogues hence they do not necessitate additional complications in their use; their advantage is that they provide a well-posed problem hence guaranty the stability of numerical schemes. These assessments are demonstrated for an all-dielectric structure; the effective susceptibility tensors are derived thanks to asymptotic analysis combined with homogenization technique and bounds for the susceptibilities entering the balance of energy are provided. Validation of the effective model is provided by means of comparison with direct numerics in two and three dimensions.

1. Introduction

Metasurfaces are smartly engineered two-dimensional structures that offer an attractive alternative to their bulk three-dimensional analogue. Because of their subwavelength thickness, they are less lossy, more compact and of relatively ease of fabrication. To describe metasurfaces, effective models have to encapsulate the microscopic distributions of currents and fields resulting in transition conditions relating the field differences at the two opposite sides of the film to their mean values. Through this homogenization process, the effects of the microscopic distributions are encoded in macroscopic parameters which can be interpreted in terms of effective surface currents and surface polarisations. In their most general form, these conditions are called General Sheet Transmission Conditions (GSTCs) [?].

2. The actual and the effective problems

2.1. The actual problem

We consider the Maxwell's equations in three-dimensions for dielectric materials with permittivities denoted $\varepsilon_0\varepsilon_r^\pm$ in the two substrates, $\varepsilon_0\varepsilon_r^{sc}$ in the scattering particles and a uniform permeability $\mu = \mu_0$. The particles are evenly distributed at the interface $x = 0$ between the two sub-

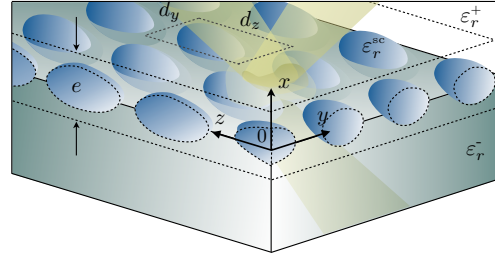


Figure 1: Actual problem of a metafilm composed by dielectric scattering particles evenly distributed at the interface between two dielectric substrates (with spacings d_y along y and d_z along z).

strates, with spacing d_y along y and d_z along z . We define $\mathcal{S} = d_y d_z$ the area of the periodic cell and $d = \sqrt{\mathcal{S}}$. In the time domain, the electric \mathbf{E} , displacement \mathbf{D} and magnetizing \mathbf{H} fields satisfy the Maxwell's equations in the actual problem (figure 1), namely

$$\begin{aligned} \text{rot}\mathbf{H} &= \frac{\partial\mathbf{D}}{\partial t}, \quad \mathbf{D} = \varepsilon_0\varepsilon_r(\mathbf{x})\mathbf{E}, \\ \text{rot}\mathbf{E} &= -\mu_0\frac{\partial\mathbf{H}}{\partial t}, \quad \text{div}\mathbf{H} = 0 \end{aligned} \quad (1)$$

where \mathbf{H} and $\mathbf{E} \times \mathbf{n}$ are continuous at the interfaces. Mathematically, $[\mathbf{H} \times \hat{\mathbf{x}}] = \mathbf{0}$ and $[\mathbf{E} \times \hat{\mathbf{x}}] = \mathbf{0}$.

2.2. The effective problem, surfacic homogenization

Using an asymptotic analysis combined with homogenization tools, it is possible to obtain an effective model in which the metasurface is replaced by the following transition conditions:

$$\begin{aligned} [\mathbf{H} \times \hat{\mathbf{x}}] &= -\varepsilon_0\frac{\partial\mathbf{P}_\parallel}{\partial t} - \nabla_\parallel M_x \times \hat{\mathbf{x}}, \\ [\mathbf{E} \times \hat{\mathbf{x}}] &= \mu_0\frac{\partial\mathbf{M}_\parallel}{\partial t} - \nabla_\parallel P_x \times \hat{\mathbf{x}}, \end{aligned} \quad (2)$$

where for any field $\mathbf{A} = \mathbf{D}, \mathbf{E}, \mathbf{H}$ we have defined the jump $[\mathbf{A}] = \mathbf{A}|_{e^+} - \mathbf{A}|_{e^-}$ and mean $\mathbf{A}_{av} = \frac{1}{2}(\mathbf{A}|_{e^+} + \mathbf{A}|_{e^-})$ operators across the metasurface's layer (figure 2). The vector $\hat{\mathbf{x}}$ is the unit vector along Ox , $\mathbf{P} = P_x\hat{\mathbf{x}} + \mathbf{P}_\parallel$, $\mathbf{M} = M_x\hat{\mathbf{x}} + \mathbf{M}_\parallel$ (with $\mathbf{P}_\parallel \cdot \hat{\mathbf{x}} = \mathbf{M}_\parallel \cdot \hat{\mathbf{x}} = 0$), and the operator $\nabla_\parallel M_x \times \hat{\mathbf{x}} = \partial_z M_x \hat{\mathbf{y}} - \partial_y M_x \hat{\mathbf{z}}$. The vectors \mathbf{P} and \mathbf{M}

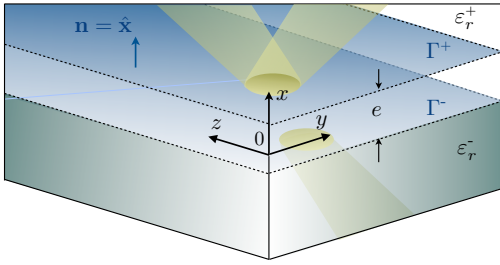


Figure 2: Effective problem in which the GSTCs (2) apply across $\Gamma^- = \{\mathbf{x} \in \Omega | x = -e^-\}$ and $\Gamma^+ = \{\mathbf{x} \in \Omega | x = e^+\}$ with $e = e^- + e^+$.

refer to the electric and magnetic polarization densities, respectively. Expectedly, we obtain zero cross-susceptibility tensors hence we have

$$\mathbf{P} = \overline{\overline{\chi}}_{ee} \tilde{\mathbf{E}}_{av}, \quad \mathbf{M} = \overline{\overline{\chi}}_{mm} \mathbf{H}_{av}, \quad (3)$$

where the subscript ‘‘av’’ refers to the average values of the fields at both sides of the interface, and where $\overline{\overline{\chi}}_{ee}$, and $\overline{\overline{\chi}}_{mm}$ are the electric-to-electric and magnetic-to-magnetic surface susceptibility tensors whose forms are

$$\overline{\overline{\chi}}_{ee} = \begin{pmatrix} \chi_{ee}^{xx} & \chi_{ee}^{xy} & \chi_{ee}^{xz} \\ \chi_{ee}^{xy} & \chi_{ee}^{yy} & \chi_{ee}^{yz} \\ \chi_{ee}^{xz} & \chi_{ee}^{yz} & \chi_{ee}^{zz} \end{pmatrix}, \quad \overline{\overline{\chi}}_{mm} = e \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

and where the 6 effective susceptibilities are defined in terms of the solutions of so-called elementary problems (see details in [1]). The average electric field has to be understood as

$$\tilde{\mathbf{E}}_{av} = \frac{1}{\varepsilon_0} D_{x,av} \hat{\mathbf{x}} + \mathbf{E}_{||av}, \quad (5)$$

since it involves the fields being continuous at the dominant order corresponding to vanishing effect of the thin metafilm.

3. Validation of the effective model

We provide below illustrations of the efficiency of the effective model by means of comparisons between the results of the effective and actual problems (in the harmonic regime in two- and three-dimensions). For the actual and the effective problems, the Finite Element Method (FEM) has been used in Comsol Multiphysics (see details in [1]).

In figure 3, we report the comparison of the actual field E_y for an ensemble of clover-like silicon particles (the source is indicated with a black circle) shown for $y < 0$ and the effective field for which the interface containing the particle has disappeared. The figure 4 show the same comparison in 3D for cuboid-shaped silicon particles with the air as unique substrate.

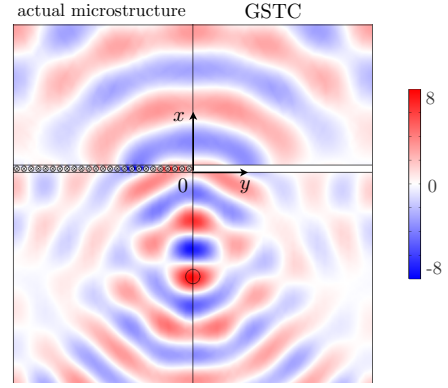


Figure 3: Real part of E_y for a layer comprising an array of clover-like silicon scattering particles. The field calculated in the actual problem is shown for $y < 0$ and the field calculated with the GSTCs is shown for $y > 0$.

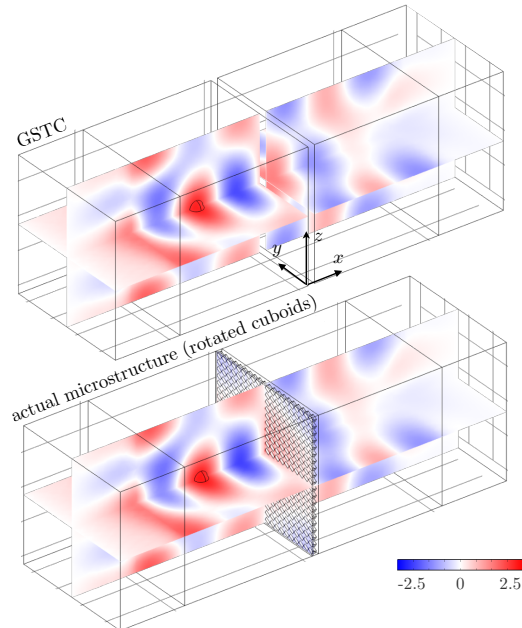


Figure 4: Real part of E_y in the effective problem (top) and in the actual problem (bottom) for the cuboid particles.

References

- [1] N. Lebbe, K. Pham, A. Maurel, Stable GSTC formulation for Maxwell’s equations, to appear in *IEEE Transactions on Antennas & Propagation* (2022).