First-order perturbation theory for electromagnetic eigenmodes

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Abstract

We present an accurate first order perturbation theory for the electromagnetic eigenmodes of a resonator, which is based on the spectral representation of the Green's dyadic. We show that when the resonator boundary is deformed, higher-order terms of the standard perturbation series can contribute to the eigenmode frequencies in first-order in the deformation depth. This is a consequence of the vectorial nature of the electromagnetic field and is technically due to the infinite degeneracy of static modes of the resonator.

1. Introduction

Resonant states (RSs) [1], also referred to as quasi-normal modes [2], are the eigenmodes of a resonator which determine its optical properties, such as the scattering cross-section or the Purcell enhancement [3]. For simple systems, such as a slab or a sphere, RSs can be found analytically [1]. For more complicated shapes they can be found numerically or via perturbative approaches [2, 3].

Here we study the role of static modes [4] in the perturbation theory for RSs, with specific focus on their contribution to the first-order approximation of the RS wavenumbers. Static modes describe the static pole of the Green's dyadic; they are infinitely degenerate and have the wavenumber eigenvalue $k = 0$. We develop a general perturbation theory that properly treats size or shape changes of the resonator. We will show that even though static modes appear only in second and higher order terms of the perturbation series, they contribute to the RS wavenumbers already in first order in the deformation depths.

2. Results

We start by considering an unperturbed system that can be solved analytically, such as a homogeneous dielectric sphere, and for simplicity we consider isotropic and non-magnetic materials only. Such system can be described by the electric Green's dyadic [5]

$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \sum_{\nu} \frac{\mathbf{E}_\nu(\mathbf{r}) \otimes \mathbf{E}_\nu(\mathbf{r}')}{k - k_\nu} = \sum_{n} \frac{\mathbf{E}_n(\mathbf{r}) \otimes \mathbf{E}_n(\mathbf{r}')}{k - k_n} + \sum_{\lambda} \frac{\mathbf{E}_\lambda(\mathbf{r}) \otimes \mathbf{E}_\lambda(\mathbf{r}')}{k}, \quad (1)$$

where $\mathbf{E}_\nu$ is the electric field of the normalized eigenmodes, $\nu = n$ labels the RSs ($k_n \neq 0$), and $\nu = \lambda$ labels the static modes ($k_\lambda = 0$). The residue of the static pole of the Green's dyadic contains a delta-like singularity [5]. Using the Green's function we can find the matrix equation that links the unperturbed modes to the perturbed ones

$$(\varkappa - k_\nu) c_\nu = -\varkappa \sum_{\nu'} V_{\nu\nu'} c_{\nu'}, \quad (2)$$

where $\varkappa$ is the eigenvalue of the perturbed system, c_ν is the expansion coefficient for the perturbed field, and

$$V_{\nu\nu'} = \int \mathbf{E}_\nu(\mathbf{r}) \cdot \Delta\varepsilon(\mathbf{r}) \mathbf{E}_{\nu'}(\mathbf{r}) \mathrm{d}\mathbf{r}\,. \quad (3)$$

This procedure is called the resonant-state expansion [1, 5] which effectively treats perturbations to all orders. From the matrix equation one can extract corrections to the eigenvalue k_n in the form of a perturbation series

$$\varkappa = k_n - k_n V_{nn} + k_n V_{nn}^2 + k_n^2 \sum_{\nu' \neq n} \frac{V_{n\nu'} V_{\nu' n}}{k_n - k_{\nu'}} + \ldots \quad (4)$$

from which it appears that $\varkappa^{(1)} = -k_n V_{nn}$ is the first-order correction to the wavenumber, being linear in the deformation depth δh. However, static modes are infinitely degenerate ($k_{\nu'} = 0$), and this allows the infinite sum in the second order term to lead to a correction linear in δh.

To extract such linear contributions from all orders, we reformulate Eq. (2) by moving all static modes into a matrix $W_{\lambda\lambda'}$ which is the inverse of $\delta_{\lambda\lambda'} + V_{\lambda\lambda'}$. Then the linear correction to $\varkappa$ can be found as

$$\varkappa^{(1)} = -k_n \tilde{V}_{nn}\,, \quad (5)$$

where

$$\tilde{V}_{nn} = V_{nn} - S_{nn}\,, \quad S_{nn} = \sum_{\lambda\lambda'} V_{n\lambda} W_{\lambda\lambda'} V_{\lambda' n}\,. \quad (6)$$

We now use a Neumann series expansion for W, then substitute the exact static pole residue of the Green's dyadic (which can written as a sum of regular and singular parts) into the series, and integrate out the delta functions from the singular part which is proportional to $\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}'$ dyadic component [5]. Finally, after some algebra, we arrive at

$$S_{nn} \approx \int \mathbf{E}_n(\mathbf{r}) \cdot \frac{[\Delta\varepsilon(\mathbf{r})]^2}{\varepsilon(r) + \Delta\varepsilon(\mathbf{r})} \left(\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \mathbf{E}_n(\mathbf{r}) \mathrm{d}\mathbf{r}\,, \quad (7)$$

where we have neglected the terms involving the regular part of the static pole and therefore not contributing in first order. The expression for S_{nn} is clearly second order in $\Delta\varepsilon$; however it is of first order in the deformation depth δh, due to the single integral. The regular part in turn leads to multiple integrals, and thus to higher order terms in δh. Subtracting S_{nn} from V_{nn} we obtain

$$\tilde{V}_{nn} = \int \mathbf{E}_{n\parallel}(\mathbf{r}) \cdot \Delta\varepsilon(\mathbf{r}) \mathbf{E}_{n\parallel}(\mathbf{r}) d\mathbf{r} + \int \mathbf{E}_{n\perp}(\mathbf{r}) \cdot \frac{\varepsilon(r)\Delta\varepsilon(\mathbf{r})}{\varepsilon(r)+\Delta\varepsilon(\mathbf{r})} \mathbf{E}_{n\perp}(\mathbf{r}) d\mathbf{r}\,, \quad (8)$$

where the subscript $\parallel$ ($\perp$) labels the vector component parallel (normal) to the surface. The correct wavenumber to first order is thus given by $\varkappa \approx k_n - k_n \tilde{V}_{nn}$. Note that for a shape perturbation in vacuum we have $\varepsilon(r) + \Delta\varepsilon(\mathbf{r}) = 1$, rendering Eqs. (5) and (8) similar to the results in Ref. [2].

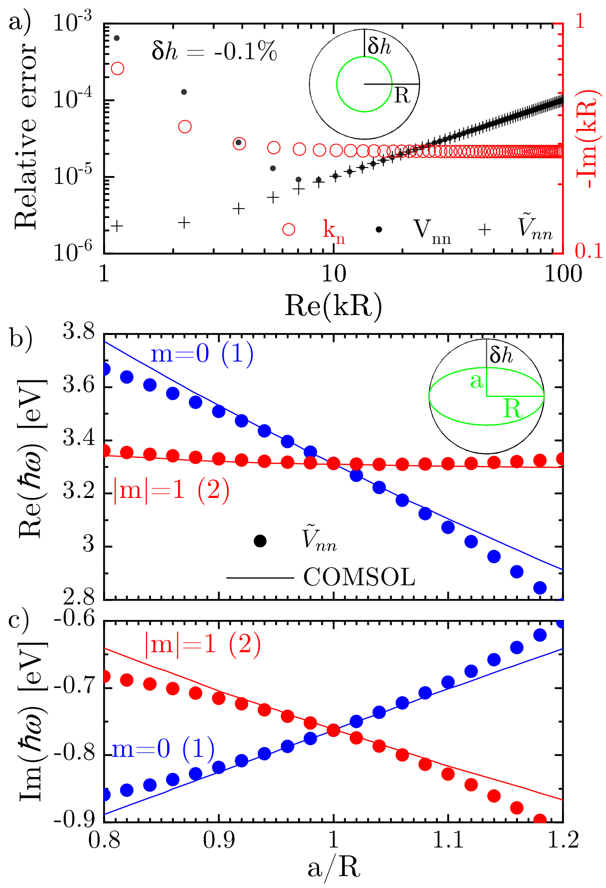

Figure 1: (a) RS wavenumbers (red) and relative errors (black) for the size perturbation of a dielectric sphere, with $\varepsilon = 4$ and radius R. (b) Real and (c) imaginary part of the surface plasmon frequency when a silver sphere is deformed into an ellipsoid, for angular momentum $l = 1$. The number in the brackets shows the degeneracy of the mode. The parameters for the silver sphere and the COMSOL data is taken from [2].

3. Discussion

When the permittivity change $\Delta\varepsilon(\mathbf{r})$ is small, for example, introducing a material change, one can neglect S_{nn} and use Eq. (3), which is a well known and tested result [3] and will not be discussed here further. When $\Delta\varepsilon(\mathbf{r})$ is large, which can be in case of a shape perturbation, the results of Eqs. (3) and (8) differ due to the contribution of the static modes S_{nn}. We show in Fig. 1(a) results for a size perturbation of a homogeneous dielectric sphere. For the modes with low wavenumbers, the relative error using Eq. (8) is a few order of magnitude smaller than using Eq. (3), showing the influence of the static modes on the first order correction. The first order results can also be extended to dispersive systems. Figures 1(b) and (c) show the real and imaginary part, respectively, for the surface plasmon mode of a silver sphere when it is being distorted into an ellipsoid as in [2]. We see good agreement between the perturbation theory results and the numerically calculated values for small changes, for both inwards ($a < R$) and outwards ($a > R$) perturbations.

4. Conclusions

We have shown that due to the static pole of the Green's dyadic higher order terms in the perturbation series can contribute in first order in the deformation depth. We have derived a general perturbation theory that can treat both material changes and shape perturbations. We have illustrated the importance of the static pole for size and shape changes of spherical resonators. This approach can be generalised to treat chirality and permeability perturbations.

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References

[1] E. A. Muljarov, W. Langbein, R. Zimmermann, Brillouin-Wigner perturbation theory in open electromagnetic systems, *Europhys. Lett.*, 92: 50010, 2010.

[2] W. Yan, P. Lalanne, M. Qui, Shape Deformation of Nanoresonator: A Quasinormal-Mode Perturbation Theory, *Physical Review Letters*, 125: 013901, 2020.

[3] S. Both, T. Weiss, Resonant states and their role in nanophotonics *Physical Review A*, 37: 013002, 2021.

[4] S. Lobanov, W. Langbein, E. A. Muljarov, Resonant-state expansion applied to three-dimensional open optical systems: Complete set of static modes *Physical Review A*, 100: 063811, 2019.

[5] E. A. Muljarov, Full electromagnetic Green's dyadic of spherically symmetric open optical systems and elimination of static modes from the resonant-state expansion *Physical Review A*, 101: 053854, 2020.